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STATIONARY MODEL OF THE GENERALIZED PRANDTL EQUATIONS AND THE PASSAGE TO THE LIMIT WITH RESPECT TO LONGITUDINAL VISCOSITY IN THE NAVIER-STOKES EQUATIONS*

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It is shown that the generalized Prandtl equations (GPE) represent a limiting case of the Navier-Stokes (NS) equations when the "longitudinal" viscosity tends to zero. An estimate for the neglected terms is obtained and a theorem of existence proved for the GPE. The theorem was established earlier /1/ for the case of homogeneous conditions.

The passage to the limit of the non-steady Euler equations is carried out in /2/ under the assumption that the vorticity vanishes on the solid surfaces. Although the assumption is not physically justified, it enables the integrals over the solid surfaces to be estimated easily.

It is well-known that the use of the Hopf truncation for the NS equations in the inhomogeneous stationary problemof flow, leads to an estimate of the nor of the velocity gradient depending exponentially on viscosity /2, 3/. We note that no such difficulty arises in the case of the non-stationary problem, nor in the Cauchy problem /4, 5/. In the first case the "smoothing" may take place with time, and in the second case there are no boundary effects at all.

The problem of flow with various boundary conditions specified in terms of the stream and Bernoulli functions, free from the above drawbacks, is studied below.

1. Formulation of the problem. The flow takes place within the square $\Omega = (0,1) \times (0,1)$. We denote the segment $x = 0$ by Γ_1 and number the remaining sides $\Gamma_{2,3,4}$ in an anticlockwise direction $\Gamma_{1,3}$ denote the inflow and outflow segments respectively, and $\Gamma_{2,4}$ are rigid walls. Introducing the Bernoulli function $H = p + {}^1/_2 (\psi_y')^2 + {}^1/_2 (\psi_x')^2 + \Pi$ (the notation is standard), we consider the system of equations

$$\begin{aligned} &(\nu_1\psi_{x^2}'' + \nu_2\psi_{y^2}'')_x' + H_y' = \psi_y'\Delta\psi + f_2, \quad 0 \leqslant \nu_1 \leqslant \nu_2 \\ &(\nu_1\psi_{x^2}'' + \nu_2\psi_{y^2}'')_y' - H_x' = -\psi_x'\Delta\psi - f_1, \quad \nu_2 > 0 \end{aligned} \tag{1.1}$$

where $f_{1,2}$ are the components of the mass force vector. When $\nu_1 = \nu_2 = \nu$, we have the NS equations and when $\nu_1 = 0$, we have the GPE. Eliminating H, we can rewrite (1.1) in the following equivalent form:

$$\nu_1 (\Delta\psi)_{x^2}'' + \nu_2 (\Delta\psi)_{y^2}'' = -\psi_x'(\Delta\psi)_y' + \psi_y'(\Delta\psi)_x' + f_{2x}' - f_{1y}'. \tag{1.2}$$

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The boundary conditions of adhesion have the form

$$\psi \mid_2 = \psi_y' \mid_{2,4} = 0, \quad \psi_x' \mid_4 = 0. \tag{1.3}$$

Two types of conditions on $\Gamma_{1,3}$ corresponds to two problems under consideration, but the Bernoulli function H is given in both cases:

$$H \mid_i = H_i(y); \; (\psi_{x^2}'' + \alpha\psi_{y^2}'') \mid_i = \gamma_i(y), \quad \alpha < 1, \; i = 1, 3 \tag{1.4}$$
(problem A)

$$H \mid_i = H_i(y); \quad -\psi_x' \mid_i = v_i(y), \quad (v_i = v_{iy}' = 0, \; y = 0, 1), \quad i = 1, 3 \text{ (problem B)}.$$

The condition $\alpha < 1$ is necessary for the problem to be Shapiro-Lopatinskii correct. The case $\alpha = -1$ corresponds to specifying the tangential stress. We note that we cannot replace the last condition of (1.3) by $\psi \mid_4 = g$, since the problem will then become overdefined and from the physical point of view the flow rate is related primarily to the values of $H_{1,3}$.

Let us introduce the notation for the norms of the right-hand sides of the problem

$$r = \| f_1 \|_{W_2^1(\Omega)}, \quad s = \| H_i \|_{C_2(\bar\Gamma_{1,3})}, \quad t = \| \gamma_i \|_{C_2(\bar\Gamma_{1,3})} . \tag{1.5}$$

The scalar product in $L_2(\Omega)$ can be written as (a, b), $\| a \| \equiv (a, a)^{1/2}$, and we use the angle brackets for the boundary integrals

$$\langle a, b \rangle_j^i = \int abd\Gamma_i - \int abd\Gamma_j .$$

We shall apply the anisotropic Sobolev space $W_p^{(\gamma_1, \gamma_2)}$, and Besov space $B_{pp}^{(\gamma_1, \gamma_2)}$, using the notation of /6/. In addition, we shall find the weight space $V_{p,\beta}^s(\Omega)$ (/7/, Sect.10) with the norm

$$\| u \| = \Big(\sum_{|\alpha| = 0}^{s} \int_\Omega r^{p(\beta - s + |\alpha|)} | D^\alpha u |^p d\Omega \Big)^{1/p} \tag{1.6}$$

useful, where r is a smooth function equal in the neighbourhood of any angle point Ω, to the distance to it. We shall regard D^n as an n-th order arbitrary differential operator.

We define the following seminorms:

$$\begin{aligned} &m_2^2 = \| \psi_{x^2}'' \|^2 + \| \psi_{xy}'' \|^2, \quad n_2^2 = \| \psi_{xy}'' \|^2 + \| \psi_{y^2}'' \|^2 \\ &m_3^2 = \| \psi_{x^3}''' \|^2, \; l_3^2 = \| \psi_{x^2y}''' \|^2 + \| \psi_{xy^2}''' \|^2, \quad n_3^2 = \| \psi_{y^3}''' \|^2 \\ &N_2^2(\alpha, \varepsilon, \nu_{1,2}) = \nu_1 (1 - \varepsilon \mid 1 - \alpha \mid 2) \| \psi_{x^2}'' \|^2 + (\nu_2 - \alpha\nu_1) \| \psi_{xy}'' \|^2 + (\nu_2 - \nu_1 \mid 1 - \alpha \mid 2\varepsilon) \| \psi_{y^2}'' \|^2 \\ &N_3^2(\alpha, \beta, \varepsilon, \nu_{1,2}) = \nu_1 \| \psi_{x^3}''' \|^2 + [\nu_1\beta - \nu_2 - \varepsilon \mid (1 + \alpha)(\nu_1\beta - \nu_2) \, 2 \mid] \| \psi_{x^2y}''' \|^2 + [\nu_2\beta - \nu_2 - \nu_1\beta - (1 - \alpha)(\nu_1\beta - \nu_2)] \| \psi_{xy^2}''' \| + [\nu_2\beta - \mid (1 + \alpha)(\nu_1\beta - \nu_2) \, 2\varepsilon \mid] \| \psi_{y^3}''' \| . \end{aligned} \tag{1.7}$$

2. Auxilliary assumptions.

We first establish the properties of smoothness.

Lemma 1 (concerning the behaviour of solutions at the angle points). Let a finite function $f \in W_2^{01}(K)$ be defined within the angle K $(x \geqslant 0, y > 0)$. Then the following estimates exist for any solution of the equation $\nu_1 (\Delta u)_{x^2}'' - \nu_2 (\Delta u)_{y^2}'' = f$ satisfying the boundary conditions:

$$(u = u_y') \mid_{y=0} = 0, \quad (u_{x^2}'' + \alpha u_{y^2}'') = (\nu_1 u_{x^2}'' + \nu_2 u_{y^2}'')_x' \mid_{x=0} = 0, \quad \alpha \leqslant 0 \tag{a}$$

$$(u = u_y') \mid_{y=0} = 0, \quad (u_x' = u_{x^3}''') \mid_{x=0} = 0 \tag{b}$$

namely:

$$\| u \|_{V^4_{p, 2-\gamma-2/p}(K)} \leqslant C(\nu_{1,2}) \| f \|_{L_p}, \quad p \in (1, \infty), \; 0 < \gamma < \varepsilon(\rho) \qquad \text{(a)} \tag{2.1}$$

$$\| u \|_{V^{4-l}_{p, 1-2/p}(K)} \leqslant C(\nu_{1,2}) \| f \|_{W_p^{l}}, \quad p \in (1, \infty), \; l = 0, 1 \qquad \text{(b)}$$

where $\rho = \nu_1 / \nu_2 \in (0, 1]$ and the monotonic function $\varepsilon(\rho)$ has the following properties:

$$\varepsilon(1) = \varepsilon_1 > 1/2 - \pi^{-1} \arccos 2\pi^{-1}, \quad \varepsilon(\rho) \to +0, \quad \rho \to 0 .$$

Proof. Making the change of variables $x = e^t\cos\varphi,\ y = e^t\sin\varphi$, writing $\mu = (1-\rho)/(1+\rho)$, replacing ∂/dt by λ and introducing the operators

$$D = d/d\varphi,\quad l = D^2 + (\lambda - 2)^2$$
$$l_\mu = (1 + \mu\cos 2\varphi)\, D^2 + 2\mu\sin 2\varphi\,(\lambda - 1)\, D + \lambda\mu\cos 2\varphi\,(2-\lambda) + \lambda^2$$

we arrive, in case (a) at the following problem in the angle K:

$$ll_\mu \hat u(\varphi) = 0,\quad \hat u(0) = \hat u'(0) = 0 \tag{2.2}$$
$$\hat u''(\pi/2) + \alpha\lambda^2\hat u(\pi/2) = \rho\hat u'''(\pi/2) + [\lambda^2 - 3\lambda(1-\rho) + 2(1-\rho)]\times \hat u'(\pi/2) = 0\,.$$

The following system of functions is the basis of the kernel of the operator of (2.2):

$$u_{1,2} = \frac{\cos}{\sin}\{\lambda\varphi\},\quad u_{3,4} = (1 + \mu\cos 2\varphi)^{\lambda/2}\frac{\cos}{\sin}\{\lambda\,\mathrm{arctg}\,(\sqrt{\rho}\,\mathrm{tg}\,\varphi)\}.$$

The dispersion equation has the form $(\alpha = -1)$

$$(1-\mu/\lambda)\left(1 - \frac{1-\sqrt{\rho}}{1+\sqrt{\rho}}\cos\lambda\pi\right) = \frac{2}{1+\rho}\,\frac{2}{1+\sqrt{\rho}}\,\rho^{(1-\lambda)/2} \tag{2.3}$$

and its limiting case $(\rho - 1)$ is identical with the equation

$$-\lambda\cos\lambda\pi = 2\lambda^2 - 3\lambda + 2 \tag{2.4}$$

generated by the basis of the kernel of the operator ll_0. In case (2.4) we find, after writing $\lambda = x + iy,\ x = {}^3/_2 + \chi$, that $\chi > \pi^{-1}\arccos 2\pi^{-1}$. A slightly more detailed inspection of the solution of (2.3) shows that the strip $\mathrm{Re}\lambda \in (2, 2+\varepsilon(\rho))$ does not contain any eigenvalues of problem (2.2). Now we can apply the theory developed in /7/, and using corollary (7.1) from /7/, we can obtain the estimate (2.1) (a) taking into account the inclusion $V^{\bullet}_{p,\beta} - L_p,\ \beta \geqslant 0$. In case (b) we arrive at relations (2.2) in which the last two conditions have been replaced by $\hat u'(\pi/2) = \hat u'''(\pi/2) = 0$. The dispersion equation has the form $\mathrm{tg}\,\lambda\pi/2 = 0$ for any ρ. This ensures the required smoothness of the solution /8/ and leads to the estimate (b) which completes the proof.

Lemma 2 (on coercivity). The differential form $\nu_1(\Delta\psi)_{x^2}'' + \nu_2(\Delta\psi)_{y^2}'',\ \psi \in W_2^4$ with conditions a) $(\psi = \psi_y')|_{2,4} = (\psi_{x^2}'' + \alpha\psi_{y^2}'')|_{1,3} = (\nu_1\psi_{x^2}'' + \nu_2\psi_{y^2}'')_x'|_{1,3} = 0$ or b) $(\psi = \psi_y')|_{2,4} = (\psi_x' = \psi_{x^3}''')|_{1,3} = 0$ is coercive in the sense that the following inequalities (using the notation (1.7)) hold:

$$1^\circ.\ -(\mathrm{grad}\,(\nu_1\psi_{x^2}'' + \nu_2\psi_{y^2}''),\ \mathrm{grad}\,\psi) \geqslant \begin{cases} N_2^2(\alpha,\ \varepsilon,\ \nu_{1,2}),\ \forall\varepsilon \in (0,\ \infty),\ \alpha \in R^1 & \text{(a)} \\ \nu_1 m_2^2 + \nu_2 n_2^2 & \text{(b)} \end{cases}$$

$$2^\circ.\ (\mathrm{grad}\,(\nu_1\psi_{x^2}'' - \nu_2\psi_{y^2}''),\ \mathrm{grad}\,(\psi_{x^2}'' - \beta\psi_{y^2}'')) \geqslant \begin{cases} N_3^2(\alpha,\ \beta,\ \varepsilon,\ \nu_{1,2}) & \text{(a)} \\ \nu_1 m_3^2 + \nu_2 l_3^2 - \nu_2\beta n_3^2 & \text{(б)} \end{cases},\ \forall\varepsilon \in (0,\ \infty),\ \alpha \in R^1,\ \beta \in [0,\ 1]$$

$$3^\circ.\ C(\nu_1,\nu_2)\,[\|\nu_1(\Delta\psi)_{x^2}'' - \nu_2(\Delta\psi)_{y^2}''\|_{W_p^l} - \nu_2\|\psi\|_{W_p^{3+l}}] \geqslant \begin{cases} \|\psi\|_{V^4_{p,\,2-\gamma-2/p}};\ l = 0;\ p \in (1,\ 2/(1-\gamma)) & \text{(a)} \\ \|\psi\|_{V^4_{p,\,1-2/p}};\ l = 0,\ 1;\ p \in (1,\ \infty) & \text{(b)} \end{cases}$$

and the quantity γ is defined in Lemma 1.

Proof. (1°). Integrating by parts we obtain, in case (a), the boundary integral $I_1 = \nu_1(1+\alpha)\langle\psi_{y^2}'',\psi_x'\rangle_1^3$. But $(\psi_{x^2}'',\psi_{y^2}'') = \langle\psi_{y^2}'',\psi_x'\rangle_1^3 + \|\psi_{xy}''\|^2$. Applying to the right-hand side of the last equation Young's inequality, we obtain the required estimate. (2°). This is proved in a similar manner. (3°). This is established with help of local graphs and subordinate decomposition of the unity (see e.g. /9/, Sect.5.1). We use Lemma 1 in the neighbourhood of the angle points, and summation completes the derivation of the estimates, which in turn proves the lemma.

Let us investigate the flow of fluid through the region in question in the Stokes approximation. We will seek the pair of functions $\{\psi,H\}$ representing the solution of the problem

$$(\nu_1\psi_{x^2}'' + \nu_2\psi_{y^2}'')_x' - H_y' = f_2,\quad (\nu_1\psi_{x^2}'' + \nu_2\psi_{y^2}'')_y' - H_x' = -f_1 \tag{2.5}$$
$$\psi|_2 = \psi_y'|_{2,4} = \psi_x'|_4 = 0\qquad \begin{cases} (\psi_{x^2}'' + \alpha\psi_{y^2}'')|_{1,3} = \gamma_{1,3}(y) & \text{(a)} \\ -\psi_x'|_{1,3} = v_{1,3}(y)\,. & \text{(b)} \end{cases}$$
$$H|_{1,3} = H_{1,3}(y)$$

We introduce the functions $H_0,\ \psi_0,\ \psi_1$ which "take" the following boundary values:

$$H_0|_{1,3}=H_{1,3},\quad (\psi''_{ax^2}+\alpha\psi''_{ay^2})|_{1,3}=\gamma_{1,3},\quad -\psi'_{bx}|_{1,3}=v_{1,3}$$
$$\psi_{a,b}=\psi'_{a,b_y}=0,\quad y=0,1\,.$$

Lemma 3 (justification of the Stokes model). Let the right-hand sides of problem (2.5)(a), (b) have the properties $f_{1,2}\in W_p^{-1+l}(\Omega)$, $H_{1,3}\in B_p^{3/2-l}(\Gamma_{1,3})$, $\gamma_{1,3}\in B_p^{3/2+l}(\Gamma_{1,3})$, $v_{1,3}\in B_p^{5/2+l}(\Gamma_{1,3})$. $l=0,1$; $\alpha\in(-1,0)$.

Then for $p\in(1,2/(1-\gamma))$ (a), $p\in(1,\infty)$ (b) a unique solution of each problem

$$\{D^3\psi,\ DH\}\in L_p(\Omega),\ \{D^4\psi,\ D^2H\}\in V^0_{p,\,2-\gamma-2/p}(\Omega)\ \text{(a)},\ \{D^4\psi,\ D^2H\}\in W_p{}^l(\Omega)\ \text{(b)},$$

exists and

$$\nu_2\|D^4\psi\|_{V^0_{p,\,2-\gamma-2/p}}+\nu_2\|\psi\|_{W_p^3}+\|D^2H\|_{V^0_{p,\,2-\gamma-2/p}}+\|H\|_{W_p^1}\leqslant C(\nu_1/\nu_2)\quad(r_1{}^0+s_1{}^0+t_1{}^0) \tag{2.6}$$

$$\nu_2\|\psi\|_{W_p^{4-l}}+\|H\|_{W_p^{2-l}}\leqslant C(\nu_1/\nu_2)\cdot(r_1{}^l+s_1{}^l+v_1{}^l) \tag{2.7}$$

where

$$r_1{}^l=\|f\|_{W_p^{1-l}},\quad s_1{}^l=\|H_i\|_{B_p^{3/2-l}},\quad t_1{}^l=\|\gamma_i\|_{B_p^{3/2+l}},\quad v_1{}^l=\|v_i\|_{B_p^{5/2+l}}\,.$$

Proof. Let us consider the more complex case (a). We shall write the solution in the form $\psi=\bar\psi+\psi_a$, $H=\bar H+H_0$, with $\|\psi_a\|_{W_p{}^4}+\|H_0\|_{W_p{}^2}\leqslant C(s_1{}^0+t_1{}^0)$. Multiplying Eqs.(2.5) scalarly by $\bar\psi_x{}'$ and $\bar\psi_y{}'$ and applying the estimate (1°) of Lemma 2, we obtain

$$N_2{}^2\leqslant|-(\bar f_2,\ \bar\psi_x{}')+(\bar f_1,\ \bar\psi_y{}')|$$

and, since for given α we can choose ε so that $N_2{}^2\geqslant C(m_2{}^2+n_2{}^2)$, it follows that

$$\|\psi\|_{W_2{}^2}\leqslant C(r_1{}^0+s_1{}^0+t_1{}^0)\,. \tag{2.8}$$

The uniqueness of the solution follows from estimate (2.8). Let $\bar\psi|_4=g$. Then $|g|\leqslant C(r_1{}^0+s_1{}^0+t_1{}^0)$. Let us write $\bar\psi=\bar{\bar\psi}+\bar\psi_0$, $\bar\psi_0|_4=g$; $\bar{\bar\psi}$ satisfies the condition (a) of Lemma 2, and the estimate 3°(a) holds for it. But the first inclusion of $V^4_{p,\,2-\gamma-2/p}\to B_p^{3+(\gamma-2/p-1)}\to W_p{}^3$ is continuous (see the end of the proof of Lemma 4), and the second inclusion is completely continuous when $p\in(1,2/(1-\gamma))$. Using the theory of linear equations /10/, the estimate for ψ and the property of compactness mentioned above, we obtain (2.6). Now the solvability follows everywhere from the fact that the solution of the homogeneous problem is trivial, which proves the lemma.

Below we shall utilize the inclusion theorems of various metrics and measurements for the spaces W, B defined in /6/.

Lemma 4 (on the estimate of the boundary integrals). Let the functions $f\in W_p{}^2(\Omega)$, g, $h\in W_p{}^1(\Omega)$, $p\in(1,\infty)$. Then everyone of these functions has a trace on the segment $(x_1=a,\ x_2\in[0,1])$ and the following estimates hold:

$$I_1=\Big(\int_0^1|g|^p|_{x_1=a}dx_2\Big)^{1/p}\leqslant C(\varepsilon)\,[k\|g\|_{L_p}^{1-1/p-\varepsilon}\|g\|_{W_p^{(1,\,0)}}^{1/p+\varepsilon}+(1-k)\|g\|_{L_q}^{1-1/p-2/q-\varepsilon}\|g\|_{W_q^1}^{2/q+\varepsilon-1/p}],\quad\forall\varepsilon,\ q,\ k: \tag{2.9}$$
$$\varepsilon>0,\quad 1<q<\infty,\quad 0>1/p-2/q-\varepsilon\geqslant-1,\quad p\geqslant q,\qquad 0\leqslant k\leqslant1$$

$$I_2=\int_0^1|fgh|\,|_{x_1=a}\,dx_2\leqslant C(\varepsilon)\|g\|^{1/2-\varepsilon}\|g\|_{W_2^{(1,\,0)}}^{1/2+\varepsilon}\|h\|^{1/4-\varepsilon}\times[k\|f\|^{1/4-\varepsilon}\|f\|_{W_2^1}^{1/4}\|f\|_{W_2^{(2,\,1)}}^{1/4+\varepsilon}\|h\|_{W_2^{(1,\,0)}}^{1/4+\varepsilon}+(1-k)\|f\|_{W_2^1}\|h\|_{W_2^1}^{1/4+\varepsilon}] \tag{2.10}$$

$$I_3=\int_0^1|gh|\,|_{x_1=a}\,dx_2\leqslant \tag{2.11}$$
$$\begin{cases}C(\varepsilon,\ \gamma)\|g\|^{\gamma-\varepsilon}\|g\|_{W_2^1}^{1+\varepsilon-\gamma}(\|Dh\|_{V^0_{1/(1-\gamma),\,\gamma}}+\|h\|_{L_{1/(1-\gamma)}}) & \text{(a)}\\ 0<\varepsilon<\gamma\leqslant1/2, & \\ C(\varepsilon)\|g\|^{1/2-\varepsilon}\|g\|_{W_2^1}^{1/2+\varepsilon}\|h\|^{1/2-\varepsilon}\|h\|_{W_2^1}^{1/2+\varepsilon} & \text{(b)}\\ 0<\varepsilon\leqslant1/2\,. & \end{cases}$$

Proof. We shall use the multiplicative inequalities for the functions defined in R^n

$$\|f\|_{B_q{}^r} \leqslant C\|f\|_{L_q}^{1-r-\varepsilon}\|f\|_{W_q^1}^{r+\varepsilon}, \quad \varepsilon>0, \quad 1-r-\varepsilon \geqslant 0, \quad 1<q<\infty. \tag{2.12}$$

This follows from inequality (7") of /6/, Sect.7.2 and the inclusion $B_p^{k+\varepsilon} \to W_p^k$, $\varepsilon>0$ /6/. Using (2.12) we readily obtain the estimate

$$\max_{\substack{x_2\in[0,1]\\ x_1=a}} |g| \leqslant C(\varepsilon)\|g\|_{L_p(\Gamma_a)}^{1-1/p-\varepsilon}\|g\|_{W_p^1(\Gamma_a)}^{1/p+\varepsilon} \tag{2.13}$$

provided that we take into account the inclusions $B_p^{\varepsilon_1+1/p}(\Gamma_a) \to H_\infty^{\varepsilon_1}(\Gamma_a) \to C_0(\overline{\Gamma}_a)$, $0<\varepsilon_1<\varepsilon$ (Sect.6.3). Now, using the estimate (2.13) and the Hölder inequality, we obtain

$$I_1 \leqslant \left\{\int_0^1 \max|g(x_1, x_2)|^p dx_2\right\}^{1/p} \leqslant C(\varepsilon)\|g\|_{L_p(\Omega)}^{1-1/p-\varepsilon}\|g\|_{W_p^{(1,0)}(\Omega)}^{1/p+\varepsilon}. \tag{2.14}$$

Moreover, $I_1 \leqslant C\|g\|_{B_p^{\varepsilon_1}(\Gamma_a)}$ and we have the inclusions

$$B_q{}^r(\Omega) \to B_p^{1/p+\varepsilon_1}(\Omega) \to B_p{}^{\varepsilon_1}(\Gamma_a), \quad 1 \geqslant r = 2/q - 1/p + \varepsilon_1 > 0.$$

Using (2.12) we obtain

$$I_1 \leqslant C(\varepsilon)\|g\|_{L_q(\Omega)}^{1-1/p-2/q-\varepsilon}\|g\|_{W_q^1(\Omega)}^{2/q+\varepsilon-1/p}, \quad 0<\varepsilon_1<\varepsilon$$

which, combined with (2.14), yields relation (2.9).

The estimate for the integral I_2 follows from the two inequalities:

$$I_2 \leqslant C(\varepsilon)\|f\|_{L_2(\Gamma_a)}^{1/2-\varepsilon}\|f\|_{W_2^1(\Gamma_a)}^{1/2+\varepsilon}\|g\|_{L_2(\Gamma_a)}\|h\|_{L_2(\Gamma_a)}$$
$$I_2 \leqslant C(\varepsilon)\|f\|_{L_{2(2-\varepsilon)/\varepsilon}(\Gamma_a)}\|g\|_{L_2(\Gamma_a)}\|h\|_{L_{2-\varepsilon}(\Gamma_a)}$$

and for the integral I_3 we have, using (2.9) $(k=0)$,

$$I_3 \leqslant \|g\|_{L_{1/\gamma}(\Gamma_a)}\|h\|_{L_{1/(1-\gamma)}(\Gamma_a)} \leqslant C(\varepsilon,\gamma)\|g\|^{\gamma-\varepsilon}\|g\|_{W_2^1}^{1+\varepsilon-\gamma}\|h\|_{B^\varepsilon_{1/(1-\gamma)}(\Gamma_a)}$$
$$(0<\varepsilon<\gamma\leqslant 1/2).$$

Let us further use inequality (9) of Sect.10.1 of /6/, and note that the weight spaces

$$\|h\|_{W^1_{1/(1-\gamma),(-\gamma)}} \leqslant C[\|Dh\|_{V^0_{1/(1-\gamma),\gamma}} + \|h\|_{L_{1/(1-\gamma)}}]$$

by definition, which proves the lemma.

3. A priori estimates (AE) of the solutions of the problem (1.1), (1.3), (1.4) (a) (b) fall into three groups for the expressions $D^2\psi$, $D^3\psi$, $D^4\psi$. Below we shall use the notation (1.5), (1.7) and begin by deriving AE I. Scalar multiplying equations (1.1) by ψ_x' and ψ_y' and combining, we obtain

$$\nu_1 m_2^2 + \nu_2 n_2^2 = \nu_1\langle\psi_{x^2}'' - \psi_{y^2}'', \psi_x'\rangle_1^3 - \langle H, \psi_y'\rangle_1^3 - (f_2, \psi_x') + (f_1, \psi_y').$$

Simple estimates using Lemma 2 (1°) yield

$$N_2^2(\alpha, \varepsilon, \nu_{1,2}) \leqslant C[\nu_1 t m_2 + (s+r)n_2]. \quad \text{(a)} \tag{3.1}$$
$$\nu_1 m_2^2 + \nu_2 n_2^2 \leqslant C(s+r)n_2''. \quad (\upsilon_{1,3}=0). \quad \text{(b)}$$

To derive AE II we perform an analogous operation, multiplying the components of the vector $\operatorname{grad}(\psi_{x^2}'' + \beta\psi_{y^2}'')$. Considering case (a) first and taking into account the inequality (2°) (a) of Lemma 2, we obtain

$$\begin{aligned} N_3^2 \leqslant{} & (\nu_1\beta + \nu_2)\langle\psi_{xy}'', \gamma'\rangle_1^3 + (f_{1y}' - f_{2x}', \psi_{x^2}'' + \beta\psi_{y^2}'') - \\ & \langle H_y', \psi_{x^2}'' + \beta\psi_{y^2}''\rangle_1^3 + \beta\langle H_x', \psi_{x^2}''\rangle_2^4 + \\ & (\psi_y'(\psi_{x^2}'' + \beta\psi_{y^2}'')_x' - \psi_x'(\psi_{x^2}'' + \beta\psi_{y^2}'')_y', \Delta\psi). \end{aligned} \tag{3.2}$$

Using the second equation of (1.1) and the properties of $f_{1,2}$ and applying the inequality (2.11) (a) when $x_1=y$, $x_2=x$, $g=\psi_{y^2}''$, $h=\psi_{y^2}'''$, $\varepsilon=\gamma/2$, we will write (this is the central stage of the derivation of AE II):

$$|\langle H_x', \psi_{y^2}''\rangle_2^4| = |\langle\psi_{y^2}''', \psi_{y^2}''\rangle| \leqslant \nu_2 C(\gamma) n_2^{\gamma/2}(l_3+n_3)^{1-\gamma/2}(\|D^4\psi\|_{V^0_{1/(1-\gamma),\gamma}} + n_3).$$

Integrating the trilinear form in (3.2) by parts and using the condition (1.4) (a), we obtain

$$\tfrac{1}{2}\langle\psi_y', [\psi_{x^2}'' - (\alpha-1)\psi_{y^2}'']^2 + (\beta-1)(\psi_{y^2}'')^2\rangle_1^3 + (\beta-1)(\psi_x'\psi_{y^2}'' - \psi_y'\psi_{x^2}''', \psi_{x^2y}''') .$$

Let us obtain estimates for the principal terms. Using the inequality (2.10) ($k = 1$), we obtain

$$|\langle\psi_y', (\psi_{y^2}'')^2\rangle| \leqslant C(\delta)\, n_2^{7/4-\delta} l_3^{1/4+\delta}$$

and taking into account (2.9) ($k = 1$), we find

$$|(\psi_{x^2y}''', \psi_x'\psi_y'')| \leqslant \|\psi_{x^2y}'''\| \left[\int_\Omega (\psi_x')^2 (\psi_{y^2}'')^2\, d\Omega\right]^{1/2} \leqslant C(\delta)\, n_2 l_3 (n_2^{1/2} l_3^{1/2} - n_2^{1/2-\delta} l_3^{1/2-\delta}) .$$

Substituting all the estimates into inequality (3.2) and neglecting higher-order terms, we obtain

$$\begin{aligned} N_3^2 \leqslant\ & \nu_2\beta C(\gamma)\, n_2^{\gamma/2} (l_3 + n_3)^{1-\gamma/2} (\|D^4\psi\|_{\overset{\circ}{V}_{1/(1-\gamma),\gamma}} + n_3) + \\ & C(\delta)\, n_2^{3/2-\delta} l_3^{1/4+\delta} (n_2^{1/4} + |1-\beta|\, l_3^{1/4}) + C(r, s, t)(n_2 + 1) . \end{aligned} \tag{3.3}$$

In the same manner we derive AE II (b):

$$\begin{aligned} \nu_1 m_3^2 + \nu_2 l_3^2 - \beta\nu_2 n_3^2 \leqslant\ & \beta\nu_2 C(\varepsilon)\, n_2^{1/2-\varepsilon} (n_3 + l_3) \|\psi\|_{W_2^3}^{1/2+\varepsilon} + \\ & C(\varepsilon)(m_2 + n_2)^{2-\varepsilon}[\beta(m_3 + l_3 + n_3)^{1-\varepsilon} + |1-\beta|(m_3 + \\ & l_3)^{1-\varepsilon}] . \end{aligned} \tag{3.4}$$

The third group of a priori estimates AE III are obtained from the estimates (2.6), (2.7) ($l = 0$) of Lemma 3, provided that we replace r_1^c by the norm in L_2 of the right-hand side of equation (1.2)

$$\begin{aligned} \left.\begin{array}{l} \|D^4\psi\|_{\overset{\circ}{V}_{1/(1-\gamma),\gamma}} \\ \|\psi\|_{W_2^4} \end{array}\right\} \leqslant\ & C(\nu_{1,2})[r + s + t + u + \\ & C(\varepsilon)(m_2 + n_2)^{1-\varepsilon}(m_3 + l_3 + n_3)^{1+\varepsilon}] . \end{aligned} \tag{3.5}$$

4. The existence of solutions and passage to the limit.

Below we shall prove the theorem of the existence of solutions of the flow problem in two formulations. We introduce a Hilbert space

$$\begin{aligned} M_\alpha = \{\psi \mid \psi \in W_2^3(\Omega),\ & \psi|_2 = \psi_x'|_4 = \psi_y'|_{2,4} = 0, \\ & (\psi_{x^2}'' - \alpha\psi_{y^2}'')|_{1,3} = 0\} \end{aligned}$$

with the norm $\|\cdot\|_{M_\alpha} = \|D^3\psi\|$ equivalent to $\|\cdot\|_{W_2^3}$. We also define a class of functions

$$V_\gamma^{(l)}(\Omega) = \{u \mid D^l u \in L_2(\Omega),\ D^{l+1} u \in \overset{\circ}{V}_{1/(1-\gamma),\gamma}(\Omega)\},\quad \gamma \in (0, 1/2]$$

Theorem 1. (the existence of solutions of the general problem). Let the right-hand sides of problem (1.1), (1.3), (1.4) (a) $H_{1,3}, \gamma_{1,3} \in C_2(\Gamma_{1,3})$, $f_{1,2} \in \overset{\circ}{W}_2^2(\Omega)$ and let the parameters satisfy the conditions $\alpha \in [-1, 0)$, $0 < \nu_1 \leqslant \nu_2$. Then its solution $\psi \in V_\gamma^{(3)}$, $H \in V_\gamma^{(1)}$ exists and the following inequalities hold:

$$\begin{aligned} & \sqrt{\nu_1\nu_2}\, m_2 + n_2 \leqslant C(r, s, t)\, \nu_2^{-1},\quad \|\psi\|_{M_\alpha} \leqslant \\ & \quad C(\gamma; r, s, t)\, \nu_1^{-(\gamma-1}\nu_2^{-1} \\ & \|D^4\psi\|_{\overset{\circ}{V}_{1/(1-\gamma),\gamma}} \leqslant C(\gamma; r, s, t)\, \nu_1^{-6/\gamma-5/2}\nu_2^{-3/2} \end{aligned} \tag{4.1}$$

where $\gamma \in (0, \varepsilon(\rho))$, and the function $\varepsilon(\rho)$ is defined in Lemma 1.

Proof. Writing the required function in the form $\psi = \psi_0 + \overline{\psi}$ (ψ_0 satisfies the conditions (1.3), (1.4) (a)), we will consider the operator $A : M_\alpha \to M_\alpha$, placing the functions $\overline{\psi}_1$ on the right-hand side of system (1.1) in one-to-one correspondence with the solution of the Stokes problem $\overline{\psi}_2$ (Lemma 3). We will show, in accordance with the Leray-Schauder principle /3/, that A is a completely continuous operator and the a priori estimate $\|\psi\|_{M_\alpha} \leqslant K$ holds for any possible solution of Eqs. (1.1) with a multiplier $\lambda \in [0, 1]$.

For any specified λ, ε can be chosen such, than when $\beta = 1$, the estimate (3.1) (a) yields

$$\sqrt{\nu_1/\nu_2}\, m_2 + n_2 \leqslant C(r, s, t)\, \nu_2^{-1} \tag{4.2}$$

and substituting the estimate (3.5) (a) into the inequality (3.3) ($\beta = 1$), we obtain

$$\nu_1 (m_3^2 + l_3^2 + n_3^2) \leqslant \nu_2/\nu_1 C(\gamma; r, s, t)\, (m_2 + n_2)^{1+\gamma/2} (m_3 + l_3 + n_3)^{2-\gamma/2} . \tag{4.3}$$

Relation (4.1) easily follows from (4.2), (4.3). But by virtue of the estimate (3.5) (a) the operator A transforms the set bounded in M_α, into a set bounded in $V_\gamma^{(3)}$, and hence compact in M_α (Lemma 3), which proves the theorem.

Theorem 2. (the existence of solutions of the NS equations). When $\nu_1 = \nu_2 = \nu$, the problem of flow (1.1), (1.3), (1.4) (a) has, under the conditions of Theorem 1, a solution with the estimates $(\forall\delta > 0)$

$$m_2 + n_2 \leqslant C(r, s, t)\,\nu^{-1}, \quad \|\psi\|_{M_\alpha} + \nu^{-1}\|H\|_{W_2^1} \leqslant C(\delta; r, s, t)\,\nu^{-3-\delta}$$
$$\|D^4\psi\|_{\overset{\circ}{V}_{2,\varepsilon_1}} + \nu^{-1}\|D^2H\|_{\overset{\circ}{V}_{2,\varepsilon_1}} \leqslant C(\delta; r, s, t)\,\nu^{-5-\delta}.$$

When ν is sufficiently large (compared with r, s, t), the solution is unique.

Proof. The above estimates are easily obtained, provided that the estimate in the space $\overset{\circ}{V}_{2,\varepsilon_1}$ (Lemma 1) is taken into account. The uniqueness can be proved using standard methods /1/.

Passing to the GPE we introduce the classes of functions

$$U(\Omega) = \{u \mid u \in W_2^2(\Omega);\ u\,|_2 = u_x'\,|_4 = u_y'\,|_{2,4} = 0,\ u_x'\,|_1 = 0\}$$
$$W(\Omega) = \{\psi \mid \psi \in W_2^2(\Omega),\ \psi_{xy}'' \in W_2^1(\Omega);\ \psi|_2 = \psi_x'|_4 = \psi_y'|_{2,4} = 0\}.$$

We shall call the pair of functions $\psi \in W(\Omega)$, $\pi \in W_2^{(1,2)}(\Omega)$ such that

$$\pi(x, y) = \int_0^y H(x, \xi)\,d\xi, \quad \pi|_{1,3} = \int_0^y H_{1,3}(\xi)\,d\xi \tag{4.4}$$

and $\forall u \in U(\Omega)$, and the following relations hold:

$$(\nu\psi_{xy^2}''' + \pi_{y^2}'', u_x') = (\psi_y'\Delta\psi + f_2, u_x')$$
$$(\nu\psi_{y^3}''' - \pi_x', u_{y^2}'') = (\psi_x'\Delta\psi + f_1, u_y') \tag{4.5}$$

the generalized solution of the problem of flow for the GPE.

Theorem 3 (existence of solutions of the GPE). Let the functions $f_{1,2}$, $H_{1,3}$ satisfy the conditions of Theorem 1. Then a pair (ψ, π) exists satisfying (4.4), (4.5) with the estimates

$$n_2 \leqslant C(r, s, t)\,\nu^{-1}, \quad l_3 \leqslant C(\delta; r, s, t)^{-5-\delta}, \quad \forall\delta > 0. \tag{4.6}$$

Equations (4.5) represent the limiting case of (1.1), and the following estimates hold for the neglected terms $(\nu_2 = \nu,\ \delta > 0)$:

$$\|\nu_1\psi_{x^3}'''\| \leqslant \sqrt{\nu_1}\,C(\delta; r, s, t)\,\nu^{-9/2-\delta}, \quad \|\nu_1\psi_{x^2y}'''\| \leqslant \nu_1 C(\delta; r, s, t)\,\nu^{-5-\delta}. \tag{4.7}$$

Proof. We shall consider the sequence $\nu_1^{(n)} \to 0$, $n \to \infty$ and another corresponding sequence of solutions represented by Theorem 1 with $\alpha = -1$, $\nu_2 = \nu$, $\nu_1 = \nu_1^{(n)}$. The inequality (3.3) $(\beta = 0)$ together with (4.2) yields

$$\sqrt{\nu_1\nu_2}\,m_3 + l_3 \leqslant C(\delta; r, s, t)\,\nu_2^{-5-\delta}$$

which readily yields the relations (4.6), (4.7).

Now we write, in place of Eqs. (1.1), $(u \in U(\Omega))$

$$\nu_1^{(n)}(\psi_{x^3}^{(n)'''}, u_x') - (\nu\psi_{xy^2}^{(n)'''} + \pi_{y^2}^{(n)''}, u_x') = (\psi_y^{(n)'}\Delta\psi^{(n)} + f_2, u_x')$$
$$\nu_1^{(n)}(\psi_{x^2y}^{(n)'''}, u_y') - (\nu\psi_{y^2}^{(n)''} - \pi_x^{(n)'}, u_{y^2}'') = -(\psi_x^{(n)'}\Delta\psi + f_1, u_y') \tag{4.8}$$

and note that by virtue of (4.6) the set $\{\psi^{(n)}\}_{n=\overline{1,\infty}}$ is bounded, and therefore weakly compact in $W(\Omega)$ (which should naturally be regarded as a Hilbert space). This implies that a sequence $\psi^{(m)}$: $\psi^{(m)} \to \psi$ exists weakly in $W(\Omega)$. Then the first terms of (4.8) will be equal to zero in the limit by virtue of the estimates (4.7), and we arrive at relations (4.5). The boundary conditions hold by virtue of the smoothness established here, and this proves the theorem.

Below we give a conditional result regarding the uniqueness of the solutions of the GPE in the class

$$E_\mu = \{\psi \mid \psi \in W_2^3(\Omega),\ \psi_y'\,|_1 > \mu y(1 - y),\ \mu > 0\}.$$

Theorem 4 (on the uniqueness of solutions of the GPE). For sufficiently large ν there exists at most one solution in the class $\psi \in E_\mu$ furnished by Theorem 3 and satisfying the supplementary condition $-\psi_x' = v_1(y) \in C_2(\bar{\Gamma}_1)$.

Proof. Writing the difference between two solutions as $\chi = \psi_1 - \psi_2$ and putting $u = \chi$, we easily obtain from (4.5)

$$\nu n_2^2 + \frac{1}{2}\left\{\langle \psi_{2y}', (\chi_x')^2\rangle^3 - 2\langle \psi_{2x}', \chi_x'\chi_y'\rangle^3 + \left\langle \frac{(\psi_{2x}')^2}{\psi_{2y}'}, (\chi_y')^2\right\rangle^3\right\} \leqslant$$
$$\frac{1}{2}\left\langle \frac{(\psi_{2x}')^2}{\psi_{2y}'}, (\chi_y')^2\right\rangle^3 + (\psi_{2x}'\chi_y' - \psi_{2x}'\chi_x', \chi_y'') -$$
$$(\chi_x', (\psi_{2x}'\chi_y')_x') + (\psi_{2xy}'', (\chi_x')^2).$$

The expression within the braces is not less than zero, and we therefore have

$$\nu n_2^2 \leqslant K(\nu) n_2^2, \quad K(\nu) \to 0, \quad \nu \to \infty$$

which completes the proof.

Theorem 5 (the classical solution of the NS equations). Under the conditions of Theorem 1 ($v_{1,3} = 0$) the problem (1.1), (1.3), (1.4) has a solution $\psi \in C_{3,\alpha}(\bar{\Omega})$, $H \in C_{1,\alpha}(\bar{\Omega})$, $\alpha \in (0, 1)$ with the estimates ($m_2 + n_2 = M_2$, $m_3 + l_3 + n_3 = M_3$, $C = C(r, s, w)$, $\varepsilon > 0$):

$$M_2 \leqslant C\nu^{-1}, \quad M_3 + \nu^{-1}\|H\|_{W_2^1} \leqslant C(\varepsilon)\nu^{-3-\varepsilon}, \quad \|\psi\|_{W_2^4} \leqslant C(\varepsilon)\nu^{-5-\varepsilon}.$$

For sufficiently large ν the solution is unique.

Proof. A priori estimates of the solutions of the problem in question follow from (3.4) taking (3.5) (b) into account. The existence of the solutions is established in the same manner as in Theorem 1, and the uniqueness as in Theorem 2. The relation $\psi \in W_2^4$ follows from (3.5) (b), in which case the estimate (2.7) when $l = 1$ and inclusion $W_2^5(\Omega) \to C_{3,\alpha}$ lead to the conclusion that the solution is classical, which completes the proof.

The results given here are obtained ion the basis of new estimate of the boundary integrals under the conditions of the flow problem, eliminating the presence of the "boundary layers" at the side cross-sections, which leads to an acceptable estimate of the norm of the velocity gradient.

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